Answers and Discussion From Problem #8, AA214

1. Subject: Transonic Potential Equations

   The Transonic Potential Equations or the Linear Cauchy-Riemann Equations are formed from the coupling of the steady compressible continuity equation of fluid dynamics

$$\partial_x \rho u + \partial_y \rho v = 0 \tag{1}$$

   and the vorticity definition

$$-\partial_x v + \partial_y u = 0 \tag{2}$$

   with vorticity $\omega = 0$ the irrotational potential assumption. Here ρ is density, u, v the Cartesian velocity components and the isentropic assumption leads us to

$$\rho = \left(1 - \frac{\gamma - 1}{2}\left(u^2 + v^2 - 1\right)\right)^{\frac{1}{\gamma-1}} \tag{3}$$

   with $\gamma = 1.4$ the ratio of specific heats. Pressure is defined as $p = \rho^\gamma$.

   The linear Cauchy-Riemann equations are recovered above and below by setting $\rho = 1$ instead of Eq 3.

   Combining Eq 1 and Eq 2 in vector form we have

$$\partial_x \mathbf{f}(\mathbf{q}) + \partial_y \mathbf{g}(\mathbf{q}) = 0 \tag{4}$$

   where

$$\mathbf{q} = \begin{pmatrix} u \\ v \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} -\rho u \\ v \end{pmatrix}, \quad \mathbf{g} = \begin{pmatrix} -\rho v \\ -u \end{pmatrix} \tag{5}$$

   Note: One approach to solving these equations is to cast them as a hyperbolic system where we solve

$$\partial_t \mathbf{q} + \partial_x \mathbf{f} + \partial_y \mathbf{g} = 0 \tag{6}$$

   (a) For Eq 6

      i. Find the flux Jacobians of $\mathbf{f}$ and $\mathbf{g}$.

      ii. Determine the eigenvalues and conditions under which the system is hyperbolic. (Hint: A system is hyperbolic if the eigenvalues of it's flux Jacobians are real.)

      ANSWER: Define $\frac{\partial \mathbf{f}}{\partial \mathbf{q}} = A$ and $\frac{\partial \mathbf{g}}{\partial \mathbf{q}} = B^n$
      The Jacobian matrices A and B are

$$A = \begin{bmatrix} -\rho + \rho^{2-\gamma} u^2 & \rho^{2-\gamma} uv \\ 0 & 1 \end{bmatrix}$$

      and

$$B = \begin{bmatrix} \rho^{2-\gamma} uv & -\rho + \rho^{2-\gamma} v^2 \\ -1 & 0 \end{bmatrix}$$

      The eigenvalues of the Jacobian matrices are interesting and shed some light on the form of the scheme's stability and formulation. The eigenvalues of A and B are

$$\lambda(A) = -\rho + \rho^{2-\gamma} u^2, \quad 1$$

$$\lambda(B) = \frac{\rho^{2-\gamma}uv \pm \sqrt{\rho^{4-2\gamma}u^2v^2 + 4\rho - 4\rho^{2-\gamma}v^2}}{2}$$

For the system to be hyperbolic the $\lambda's$ must be real. Since $\rho > 0$, this implies that

$$\rho^{4-2\gamma}u^2v^2 + 4\rho \geq 4\rho^{2-\gamma}v^2$$

Discussion: We can examine this condition in light of assumptions on the values of u and v. If $v = 0$ the condition is always satisfied, if $u = 0$ then $\rho^{1-\gamma}v^2 < 1$ which for $0.2 \leq \rho \leq 2$ (quite a large range of ρ) implies that $0 < |v| < 1.9$, a reasonable range. What has not been discussed here, is that the full potential equations are limited to weak shock waves, which translates in this case to $|u|, |v| \leq 2.0$, as a general rule.

(b) The fluxes of the Euler equations are homogeneous of degree 1.

i. Are the above fluxes $\mathbf{f}$ and $\mathbf{g}$ homogeneous of degree 1?, degree n?.

ii. If we replace Eq 3 with $\rho = 1$, what can be said about the properties of the system?

The fluxes in Eq. 4 are not homogeneous because of the nonlinear nature of Eq. 3.

$$\mathbf{f}(\alpha\mathbf{q}) = \begin{pmatrix} -\left(1 - \frac{\gamma-1}{2}\alpha^2\left(u^2+v^2-1\right)\right)^{\frac{1}{\gamma-1}} u \\ \alpha v \end{pmatrix} \neq \alpha\mathbf{f}(\mathbf{q})$$

This does not satisfy the homogenuity property, for any n.

For $\rho = 1$,

$$\mathbf{f}(\alpha\mathbf{q}) = \begin{pmatrix} -\alpha u \\ \alpha v \end{pmatrix} = \alpha \begin{pmatrix} -u \\ v \end{pmatrix} = \alpha\mathbf{f}(\mathbf{q})$$

and

$$\mathbf{g}(\alpha\mathbf{q}) = \begin{pmatrix} -\alpha v \\ -\alpha u \end{pmatrix} = \alpha \begin{pmatrix} -v \\ -u \end{pmatrix} = \alpha\mathbf{g}(\mathbf{q})$$

The system is linear and homogeneous of degree 1.

(a) Subject: Splitting/Factorization

i. *Space vector definition* The space vector for the natural ordering is

$$u^{(n)} = [u_1\ u_2\ u_3\ u_4\ u_5\ u_6\ u_7\ u_8]^T$$

ii. The space vector for an odd-even ordering is

$$u^{(*)} = [u_1\ u_3\ u_5\ u_7\ u_2\ u_4\ u_6\ u_8]^T$$

iii. P_{n*} is the permutation matrix that permutes the odd-even ordering to the natural ordering, and P_{*n} is the permutation matrix that permutes the natural ordering to the odd-even ordering. Simply put

$$u^{(n)} = P_{n*}u^{(*)}$$
$$u^{(*)} = P_{*n}u^{(n)}$$

The permutation matrices are easily deduced

$$P_{n*} = \begin{bmatrix} 1&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&1 \end{bmatrix} \quad \text{and} \quad P_{*n} = \begin{bmatrix} 1&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&1 \end{bmatrix}$$

Note that these permutation matrices are the transpose of each other and the inverse of each other.

iv. $A^{(n)}$ is a simple tridiagonal matrix

$$A^{(n)} = \frac{\nu}{\Delta x^2}\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

Premultiplying the generic ODE in naturally ordered form by P_{*n} gives

$$P_{*n}\frac{du^{(n)}}{dt} = P_{*n}A^{(n)}u^{(n)} + P_{*n}f$$

Since the permutation matrices are not functions of time, P_{n*} can be brought inside the time derivative on the *LHS* of the above equation to give

$$\frac{dP_{*n}u^{(n)}}{dt} = P_{*n}A^{(n)}u^{(n)} + P_{*n}f$$

Noting that $P_{n*}P_{*n} = I$, and that multiplying a term by the identity matrix leaves the term unchanged, the above equation can be written as

$$\frac{dP_{*n}u^{(n)}}{dt} = P_{*n}A^{(n)}P_{n*}P_{*n}u^{(n)} + P_{*n}f$$

Or more succintly as

$$\frac{du^{(*)}}{dt} = P_{*n}A^{(n)}P_{n*}u^{(*)} + P_{*n}f$$

Comparing with the odd-even version of the ODE

$$\frac{du^{(*)}}{dt} = A^{(*)}u^{(*)} + g$$

it readily follows that $A^{(*)} = P_{*n}A^{(n)}P_{n*}$ and $g = P_{*n}f = 0$. So the odd-even ordered spatial matrix operator is

$$A^{(*)} = \frac{\nu}{\Delta x^2}\begin{bmatrix} -2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -2 \end{bmatrix}$$

v. In an implicit Euler time differencing formulation of the generic ODE the time derivative is first order accurate, and the spatial term, Au, is evaluated at time $n+1$.

$$\frac{du}{dt} = \frac{1}{h}(u_{n+1} - u_n) + O(h) = Au_{n+1}$$

or, equivalently,

$$[I - hA]u_{n+1} = u_n + O(h^2)$$

Subtracting $[I - hA]u_n$ from both sides, and noting that $\Delta u_n \equiv u_{n+1} - u_n$, gives

$$[I - hA]\Delta u_n = hAu_n + O(h^2)$$

This is the delta form of the implicit algorithm. For the naturally ordered form the implicit matrix operator is a simple tridiagonal matrix; for the odd-even form it's a banded matrix with a bandwidth of nine.

(b) *System definition*

i. Define the odd and even matrix operators as follows

$$A^{(o)} = \frac{\nu}{\Delta x^2}\begin{bmatrix} -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A^{(e)} = \frac{\nu}{\Delta x^2}\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

The resulting ODE is

$$\frac{du^{(*)}}{dt} = \left[A^{(o)} + A^{(e)}\right]u^{(*)}$$

ii. Define the diagonal matrix, D, and the upper matrix, U, as

$$D = \frac{\nu}{\Delta x^2}\begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \quad \text{and} \quad U = \frac{\nu}{\Delta x^2}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

It readily follows that

$$A^{(o)} = \left[\begin{array}{c|c} D & 0_4 \\ \hline U & 0_4 \end{array}\right] \quad \text{and} \quad A^{(e)} = \left[\begin{array}{c|c} 0_4 & U^T \\ \hline 0_4 & D \end{array}\right]$$

iii. Results from problem 1a 2a are easily adapted to give

$$\left[I - hA^{(o)} - hA^{(e)}\right]u_{n+1}^{(*)} = u_n^{(*)} + O(h^2)$$

and in delta form

$$\left[I - hA^{(o)} - hA^{(e)}\right]\Delta u_n^{(*)} = \left[hA^{(o)} + hA^{(e)}\right]u_n^{(*)} + O(h^2)$$

and in factored delta form

$$\left[I - hA^{(o)}\right]\left[I - hA^{(e)}\right]\Delta u_n^{(*)} = \left[hA^{(o)} + hA^{(e)}\right]u_n^{(*)} + O(h^2)$$

Since $h^2 A^{(o)} A^{(e)} \Delta u_n^{(*)}$ is $O(h^3)$, the error term in the factored form remains $O(h^2)$.

iv. The odd and even matrix factors are

$$\left[I - hA^{(o)}\right] = \left[\begin{array}{c|c} I_4 - hD & 0_4 \\ \hline -hU & I_4 \end{array}\right] \quad \text{and} \quad \left[I - hA^{(e)}\right] = \left[\begin{array}{c|c} I_4 & -hU^T \\ \hline 0_4 & I_4 - hD \end{array}\right]$$

So clearly the odd matrix factor is a lower triangular matrix, and the even matrix factor is an upper triangular matrix. The factoring produced an LU decomposition of the implicit matrix operator for the odd-even ordering of the system without degrading the order of accuracy. The system may now be solved by simple forward and backward substitutions: Let

$$\tilde{u}_{n+1}^{(*)} = U u_{n+1}^{(*)} = \left[I - hA^{(e)}\right] u_{n+1}^{(*)}$$

then

$$L\tilde{u}_{n+1}^{(*)} = \left[I - hA^{(o)}\right] \tilde{u}_{n+1}^{(*)} = \left[hA^{(o)} + hA^{(e)}\right] u_n^{(*)}$$

$\tilde{u}_{n+1}^{(*)}$ may be solved for by forward substitution, and then $u_{n+1}^{(*)}$ can be solved for by backward substitution. This solution process offers a considerable savings, in terms of multiplication and division operations count, over the direct inversion of the resulting system from problem 1(a)v. Direct inversion using Gaussian elimination for an $m \times m$ nonsymmetric fully populated matrix requires $m(2m-1)(2m+1)/3$ multiplication and division operations, whereas forward and backward substitutions each require $m(m+1)/2$ multiplication and division operations.